

A GENERALIZATION OF STEINBERG'S CROSS-SECTION

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INTRODUCTION

0.1. Let G be a connected semisimple algebraic group over an algebraically closed field. Let B, B^- be two opposed Borel subgroups of G with unipotent radicals U, U^- and let $T = B \cap B^-$, a maximal torus of G . Let NT be the normalizer of T in G and let $W = NT/T$ be the Weyl group of T , a finite Coxeter group with length function l . For $w \in W$ let \dot{w} be a representative of w in NT . The following result is due to Steinberg [St, 8.9] (but the proof in *loc.cit.* is omitted): if w is a Coxeter element of minimal length in W then (i) the conjugation action of U on $U\dot{w}U$ has trivial isotropy groups and (ii) the subset $(U \cap \dot{w}U^-\dot{w}^{-1})\dot{w}$ meets any U -orbit on $U\dot{w}U$ in exactly one point; in particular, (iii) the set of U -orbits on $U\dot{w}U$ is naturally an affine space of dimension $l(w)$.

More generally, assuming that w is any elliptic element of W of minimal length in its conjugacy class, it is shown in [L3] that (i) holds and, assuming in addition that G is of classical type, it is shown in [L5] that (iii) holds. In this paper we show for any w as above and any G that (ii) (and hence (iii)) hold, see Theorem 3.6(ii) (actually we take \dot{w} of a special form but then the result holds in general since any representative of w in NT is of the form $t\dot{w}t^{-1}$ for some $t \in T$). We also prove analogous statements in some twisted cases, involving an automorphism of the root system or a Frobenius map (see 3.6) and a version over \mathbf{Z} of these statements using the results in [L2] on groups over \mathbf{Z} . Note that the proof of (ii) given in this paper uses (as does the proof of (i) in [L3]) a result in [GP, 3.2.7] and a weak form of the existence of "good elements" [GM] in an elliptic conjugacy class in W which (for exceptional types) rely on a computer.

0.2. Let w be an elliptic element of W which has minimal length in its conjugacy class C and let γ be the unipotent class of G attached to C in [L3]. Recall that γ has codimension $l(w)$ in G . As an application of our results we construct (see 4.2(a)) a closed subvariety Σ of G isomorphic to an affine space of dimension $l(w)$ such that $\Sigma \cap \gamma$ is a finite set with a transitive action of a certain finite group whose order is divisible only by the bad primes of G . In the case where C is

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the Coxeter class, Σ reduces to Steinberg's cross section [St] which intersects the regular unipotent class in G in exactly one element.

0.3. Recently, A. Sevostyanov [Se] proved statements similar to (i),(ii),(iii) in 0.1 for a certain type of Weyl group elements assuming that the ground field is \mathbf{C} . It is not clear to us what is the relation of the Weyl group elements considered in [Se] with those considered in this paper.

0.4. The following (unpublished) example of N. Spaltenstein, dating from the late 1970's, shows that the statement (i) (for Coxeter elements) in 0.1 can be false if the assumption of minimal length is dropped: the elements

$$\begin{aligned} \dot{w} &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & u_x &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & x & 0 & x \\ 0 & 0 & 1 & x & 0 & x \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\ y &= \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

of $GL_6(\mathbf{C})$ (with $x \in \mathbf{C}$) satisfy $u_x y \dot{w} u_x^{-1} = y \dot{w}$; hence if U is the group of upper triangular matrices in $GL_6(\mathbf{C})$ then in the conjugation action of U on $U \dot{w} U$, the isotropy group of $y \dot{w}$ contains the one parameter group $\{u_x; x \in \mathbf{C}\}$.

1. POLYNOMIAL MAPS OF AN AFFINE SPACE TO ITSELF

1.1. Let \mathcal{C} be the class of commutative rings with 1. Let N be an integer ≥ 1 . A family $(f_A)_{A \in \mathcal{C}}$ of maps $f_A : A^N \rightarrow A^N$ is said to be *polynomial* if there exist (necessarily unique) polynomials with integer coefficients

$$f_1(X_1, \dots, X_N), \dots, f_N(X_1, \dots, X_N)$$

in the indeterminates X_1, \dots, X_N such that for any $A \in \mathcal{C}$, f_A is the map

$$(a_1, \dots, a_N) \mapsto (f_1(a_1, \dots, a_N), \dots, f_N(a_1, \dots, a_N)).$$

For such a family we define for any $A \in \mathcal{C}$ an A -algebra homomorphism $f_A^* : A[X_1, \dots, X_N] \rightarrow A[X_1, \dots, X_N]$ by $f_A^*(X_i) = f_i(X_1, \dots, X_N)$ for all $i \in [1, N]$ (here we view f_i as an element of $A[X_1, \dots, X_N]$ using the obvious ring homomorphism $\mathbf{Z} \rightarrow A$). We have the following result.

Proposition 1.2. *Assume that $f_A : A^N \rightarrow A^N$ ($A \in \mathcal{C}$) is a polynomial family such that f_A is injective for any $A \in \mathcal{C}$. Then:*

(i) f_A is bijective in the following cases: (a) A is finite; (b) $A = \mathbf{Z}_p$, the ring of p -adic integers (p is a prime number); (c) A is a perfect field; (d) A is the ring of rational numbers which have no p in the denominator (p is a prime number); (e) $A = \mathbf{Z}$.

(ii) If A is an algebraically closed field then f_A is an isomorphism of algebraic varieties.

We prove (i). In case (a), A^N is a finite set and the result follows.

Assume that A is as in (b). For any $s \geq 1$ let $A_s = \mathbf{Z}/p^s\mathbf{Z}$ (a finite ring) and let $l_s : A \rightarrow A_s$ be the obvious homomorphism. Let $\xi \in A^N$. Let $\xi_s = l_s^N(\xi) \in A_s^N$. Using (a) for A_s we set $\xi'_s = f_{A_s}^{-1}(\xi_s) \in A_s^N$. Let $\xi' \in A^N$ be the unique element such that for any $s \geq 1$, the image of ξ' under $l_s^N : A^N \rightarrow A_s^N$ is equal to ξ'_s . Let $\tilde{\xi} = f_A(\xi')$. Then for any $s \geq 1$, $\xi, \tilde{\xi}$ have the same image under $l_s^N : A^N \rightarrow A_s^N$. Hence $\tilde{\xi} = \xi$. Thus f_A is surjective, as desired.

Assume that A is as in (c). Let A' be an algebraic closure of A . By [BR] (see also [Ax], [G1, 10.4.11]), $f_{A'}$ is bijective. Let $\xi \in A^N$. Since $A^N \subset A'^N$, we can view ξ as an element of A'^N so that $\xi' = f_{A'}^{-1}(\xi) \in A'^N$ is defined. For any $\gamma \in \text{Gal}(A'/A)$ we have

$$f_{A'}(\gamma(\xi')) = \gamma(f_{A'}(\xi')) = \gamma(\xi) = \xi = f_{A'}(\xi').$$

(The obvious action of γ on A'^N is denoted again by γ .) Using the injectivity of $f_{A'}$ we deduce that $\gamma(\xi') = \xi'$. Since this holds for any γ and A is perfect, it follows that $\xi' \in A^N$. We have $f_A(\xi') = \xi$. Thus f_A is surjective, as desired.

Assume that A is as in (d). Let $A_0 = \mathbf{Q}_p$, the field of p -adic numbers. We can view A as the intersection of two subrings of A_0 , namely $A_1 = \mathbf{Q}$ and $A_2 = \mathbf{Z}_p$. Now f_{A_i} is bijective for $i = 0, 1, 2$ by (b), (c). Let $\xi \in A^N$. For $i = 0, 1, 2$ we set $\xi'_i = f_{A_i}^{-1}(\xi) \in A_i^N$. Clearly, $\xi'_1 = \xi'_0 = \xi'_2$ hence $\xi'_0 \in A^N$ and $f_A(\xi'_0) = \xi$. Thus f_A is surjective, as desired.

Assume that $A = \mathbf{Z}$. We can view A as $\cap_p A_p$ where p runs over the set of prime numbers and A_p is the ring denoted by A in (d); the intersection is taken in the field \mathbf{Q} . Now f_{A_p} is bijective for any p (see (d)) and $f_{\mathbf{Q}}$ is bijective by (c). Let $\xi \in A^N$. We set $\xi'_p = f_{A_p}^{-1}(\xi) \in A_p^N$ and $\xi' = f_{\mathbf{Q}}^{-1}(\xi) \in \mathbf{Q}^N$. Clearly, $\xi'_p = \xi'$ for all p . Hence $\xi' \in A^N$ and $f_A(\xi') = \xi$. Thus f_A is surjective, as desired. This proves (i).

Now assume that A is as in (ii). Since $f_{A_1} : A_1^N \rightarrow A_1^N$ is bijective for any algebraically closed field A_1 (see (i)), it is enough to show that the morphism f_A is étale (see [G2, 17.9.1]). Let $A' = A \oplus A$ regarded as an A -algebra with multiplication $(a, b)(a', b) = (ab, ab' + a'b)$. The unit element is $1 = (1, 0)$. We set $T = (0, 1)$. Then $(a, b) = a + bT$ and $T^2 = 0$. Then $f_{A'}$ is defined. There exist

polynomials $f_k(X_1, \dots, X_N)$ ($k \in [1, N]$) with coefficients in \mathbf{Z} such that

$$\begin{aligned} & f_{A'}(a_1 + b_1 T, \dots, a_N + b_N T) \\ &= (f_1(a_1 + b_1 T, \dots, a_N + b_N T), \dots, f_N(a_1 + T, \dots, a_N + T b_N)) \\ &= (f_1(a_*) + \sum_{k=1}^N \frac{\partial f_1}{\partial X_k}(a_*) b_k T, \dots, f_N(a_*) + \sum_{k=1}^N \frac{\partial f_N}{\partial X_k}(a_*) b_k T) \end{aligned}$$

for any $a_* = (a_1, \dots, a_N) \in A^N$, $b_* = (b_1, \dots, b_N) \in A^N$. Since $f_{A'}$ is injective, we see that for any $a_* \in A^N$, the (linear) map $A^N \rightarrow A^N$ given by

$$b_* \mapsto \left(\sum_{k=1}^N \frac{\partial f_1}{\partial X_k}(a_*) b_k, \dots, \sum_{k=1}^N \frac{\partial f_N}{\partial X_k}(a_*) b_k \right)$$

is injective. It follows that for any $a_* \in A^N$, the $N \times N$ -matrix with (j, k) -entries $\frac{\partial f_j}{\partial X_k}(a_*)$ is nonsingular. This shows that f_A is étale. This proves (ii). The proposition is proved.

Proposition 1.3. *Assume that $f_A : A^N \rightarrow A^N$ ($A \in \mathcal{C}$) and $f'_A : A^N \rightarrow A^N$ ($A \in \mathcal{C}$) are two polynomial families such that $f'_A f_A = 1$ for all $A \in \mathcal{C}$. Then for any $A \in \mathcal{C}$, f_A^* is an A -algebra isomorphism and $f_A : A^N \rightarrow A^N$ is bijective.*

Since $f'_A f_A = 1$ we have $f_A^* f_A'^* = 1$ and f_A is injective for any A . Using 1.2 we see that $f_{\mathbf{Z}}$ is bijective hence $f_{\mathbf{Z}} f'_{\mathbf{Z}} = 1$. Let $\xi_A = f_A f'_A$. Then $(\xi_A)_{A \in \mathcal{C}}$ is a polynomial family and $\xi_{\mathbf{Z}} = 1$. Thus $\xi_{\mathbf{Z}}^*(X_i) = X_i$ for any i (there is at most one element of $\mathbf{Z}[X_1, \dots, X_N]$ with prescribed values at any $(x_1, \dots, x_N) \in \mathbf{Z}^N$). We see that the polynomials with integer coefficients which define ξ are X_1, \dots, X_N . It follows that $\xi_A = 1$ for any $A \in \mathcal{C}$; hence $f_A f'_A = 1$ and f_A is a bijection. Also, since $\xi_A^* = 1$ we have $f_A'^* f_A^* = 1$ for any A hence f_A^* is an isomorphism. The proposition is proved.

2. REDUCTIVE GROUPS OVER A RING

2.1. We fix a root datum \mathcal{R} as in [L1, 2.2]. This consists of two free abelian groups of finite type Y, X with a given perfect pairing $\langle, \rangle : Y \times X \rightarrow \mathbf{Z}$ and a finite set I with given imbeddings $I \rightarrow Y$ ($i \mapsto i$) and $I \rightarrow X$ ($i \mapsto i'$) such that $\langle i, i' \rangle = 2$ for all $i \in I$ and $\langle i, j' \rangle \in -\mathbf{N}$ for all $i \neq j$ in I ; in addition, we are given a symmetric bilinear form $\mathbf{Z}[I] \times \mathbf{Z}[I] \rightarrow \mathbf{Z}$, $\nu, \nu' \mapsto \nu \cdot \nu'$ such that $i \cdot i \in 2\mathbf{Z}_{>0}$ for all $i \in I$ and $\langle i, j' \rangle = 2i \cdot j / i \cdot i$ for all $i \neq j$ in I . We assume that the matrix $M = (i \cdot j)_{i, j \in I}$ is positive definite.

Let W be the (finite) subgroup of $\text{Aut}(X)$ generated by the involutions $s_i : x \mapsto \langle i, x \rangle i'$ ($i \in I$). For $i \neq j$ in I let $n_{i, j} = n_{j, i}$ be the order of $s_i s_j$ in W . Note that W is a (finite) Coxeter group with generators $S := \{s_i; i \in I\}$; let $l : W \rightarrow \mathbf{N}$ be the standard length function. Let w_I be the unique element of maximal length of W . For $J \subset I$ let W_J be the subgroup of W generated by $\{s_i; i \in J\}$. Let \mathcal{X} be the set of all sequences i_1, i_2, \dots, i_n in I such that $s_{i_1} s_{i_2} \dots s_{i_n} = w_I$ and $l(w_I) = n$.

2.2. Now (until the end of 2.10) we fix $A \in \mathcal{C}$. Let $\dot{\mathbf{U}}_A$ be the A -algebra attached to \mathcal{R} and to A (with $v = 1$) in [L1, 31.1.1], where it is denoted by ${}_A\dot{\mathbf{U}}$. As in *loc.cit.* we denote the canonical basis of the A -module $\dot{\mathbf{U}}_A$ by $\dot{\mathbf{B}}$. For $a, b, c \in \dot{\mathbf{B}}$ we define $m_{a,b}^c \in A$, $\hat{m}_c^{a,b} \in A$ as in [L2, 1.5]; in particular we have $ab = \sum_{c \in \dot{\mathbf{B}}} m_{a,b}^c c$. Let $\hat{\mathbf{U}}_A$ be the A -module consisting of all formal linear combinations $\sum_{a \in \dot{\mathbf{B}}} n_a a$ with $n_a \in A$. There is a well defined A -algebra structure on $\hat{\mathbf{U}}_A$ such that

$$\left(\sum_{a \in \dot{\mathbf{B}}} n_a a\right) \left(\sum_{b \in \dot{\mathbf{B}}} \tilde{n}_b b\right) = \sum_{c \in \dot{\mathbf{B}}} r_c c$$

where $r_c = \sum_{(a,b) \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}} m_{a,b}^c n_a \tilde{n}_b$. (See [L2, 1.11].) It has unit element $1 = \sum_{\zeta \in X} 1_\zeta$ where for any $\zeta \in X$ the element $1_\zeta \in \dot{\mathbf{B}}$ is defined as in [L1, 31.1.1]. Let $\epsilon : \hat{\mathbf{U}}_A \rightarrow A$ be the algebra homomorphism given by $\sum_{a \in \dot{\mathbf{B}}} n_a a \mapsto n_{1_0}$. We identify $\dot{\mathbf{U}}_A$ with the subalgebra of $\hat{\mathbf{U}}_A$ consisting of finite A -linear combinations of elements in $\dot{\mathbf{B}}$.

Let \mathbf{f}_A be the A -algebra with 1 associated to M and A (with $v = 1$) in [L1, 31.1.1], where it is denoted by ${}_A\mathbf{f}$. Let \mathbf{B} be the canonical basis of the A -module \mathbf{f}_A (see [L1, 31.1.1]). For $i \in I, c \in \mathbf{N}$, the element $\theta_i^{(c)}$ (see [L1, 1.4.1, 31.1.1]) of \mathbf{f}_A is contained in \mathbf{B} . Let $(x \otimes x') : u \mapsto x^- u x'^+$ be the $\mathbf{f}_A \otimes_A \mathbf{f}_A^{\text{opp}}$ -module structure on $\dot{\mathbf{U}}_A$ considered in [L1, 31.1.2]; we write $u x'^+$ instead of $1^- u x'^+$ and $x^- u$ instead of $x^- u 1^+$.

The elements $\{1_\zeta b^+; b \in \mathbf{B}, \zeta \in X\}$ (resp. $\{b^- 1_\zeta; b \in \mathbf{B}, \zeta \in X\}$) of $\dot{\mathbf{U}}_A$ are distinct and form a subset of $\dot{\mathbf{B}}$. Let $\hat{\mathbf{U}}_A^+$ (resp. $\hat{\mathbf{U}}_A^-$) be the A -submodule of $\hat{\mathbf{U}}_A$ consisting of elements

$$\sum_{b \in \mathbf{B}, \zeta \in X} n_b (1_\zeta b^+) \text{ (resp. } \sum_{b \in \mathbf{B}, \zeta \in X} n_b (b^- 1_\zeta))$$

with $n_b \in A$.

2.3. As in [L2, 4.1], let G_A be the set of all $\sum_{a \in \dot{\mathbf{B}}} n_a a \in \hat{\mathbf{U}}_A$ such that $n_{1_0} = 1$ and $\sum_{c \in \dot{\mathbf{B}}} \hat{m}_c^{a,b} n_c = n_a n_b$ for all $a, b \in \dot{\mathbf{B}}$ (the last sum is finite by [L2, 1.16]). Note that G_A is a subgroup of the group of invertible elements of the algebra $\hat{\mathbf{U}}_A$. As in [L2, 4.2] we set $U_A = G_A \cap \hat{\mathbf{U}}_A^+$, $U_A^- = G_A \cap \hat{\mathbf{U}}_A^-$ (in *loc.cit.* U_A, U_A^- are denoted by $G_A^{>0}, G_A^{<0}$). Let T_A the set of elements of G_A of the form $\sum_{\lambda \in X} n_\lambda 1_\lambda$ with $n_\lambda \in A$. As shown in *loc.cit.*, U_A, U_A^-, T_A are subgroups of G_A . We note that

(a) *multiplication in G_A defines an injective map $U_A^- \times T_A \times U_A \rightarrow G_A$.*

This statement appears in [L2, 4.2(a)] but the line

" $\hat{\mathbf{U}}_A^- \cap \hat{\mathbf{U}}_A^{>0} = \{1\}$. Thus, $\xi'_3 \xi_3^{-1} = 1$ so that $\xi'_3 = \xi_3$."

in the proof in *loc. cit.* should be replaced by:

" $\hat{\mathbf{U}}_A^- \cap \hat{\mathbf{U}}_A^{>0} = \{a1; a \in A\}$. Thus, $\xi'_3 \xi_3^{-1} = a1$ for some $a \in A$. Since $\epsilon(\xi'_3 \xi_3^{-1}) = 1$, we have $a = 1$ so that $\xi'_3 = \xi_3$."

From (a) we deduce that

(b) $U_A \cap U_A^- = \{1\}$.

2.4. For any $i \in I, h \in A$ we set

$$x_i(h) = \sum_{c \in \mathbf{N}, \lambda \in X} h^c 1_\lambda \theta_i^{(c)+} \in \hat{U}_A,$$

$$y_i(h) = \sum_{c \in \mathbf{N}, \lambda \in X} h^c \theta_i^{(c)-} 1_\lambda \in \hat{U}_A.$$

By [L2, 1.18(a)] we have $x_i(h) \in U_A, y_i(h) \in U_A^-$. For any $i \in I$ we set

$$\dot{s}_i = \sum_{a \in \mathbf{N}, b \in \mathbf{N}, \lambda \in X; \langle i, \lambda \rangle = a+b} (-1)^a \theta_i^{(a)-} 1_\lambda \theta_i^{(b)+} \in \hat{U}_A.$$

(Note that $\theta_i^{(a)-} 1_\lambda \theta_i^{(b)+} \in \dot{\mathbf{B}}$ whenever $a + b = \langle i, \lambda \rangle$.) By [L2, 2.2(e)] we have $\dot{s}_i \in G_A$. (In *loc.cit.*, \dot{s}_i is denoted by $s_{i,1}''$.) From [L2, 2.4] we see that if $i, j \in I, i \neq j$, then

(a) $\dot{s}_i \dot{s}_j \dot{s}_i \cdots = \dot{s}_j \dot{s}_i \dot{s}_j \cdots$ in G_A (both products have $n_{i,j}$ factors).

For $i \in I$ we have

(b) $\dot{s}_i^2 = t_i(-1)$

where $t_i(-1) = \sum_{\lambda \in X} (-1)^{\langle i, \lambda \rangle} 1_\lambda \in T_A$ normalizes U_A, U_A^- . (See [L2, 2.3(b), 4.4(a), 4.3(a)].)

For $i \in I, h \in A$ we have

(c) $\dot{s}_i^{-1} x_i(h) \dot{s}_i = y_i(-h)$.

This follows from the definitions using [L2, 2.3(c)].

For any $w \in W$ we set $\dot{w} = \dot{s}_{i_1} \dot{s}_{i_2} \cdots \dot{s}_{i_k} \in G_A$ where i_1, \dots, i_k in I are such that $s_{i_1} s_{i_2} \cdots s_{i_k} = w, k = l(w)$. This is well defined, by (a).

2.5. We have the following result (see [L2, 4.7(a)]):

(a) Let $w \in W$ and let $i \in I$ be such that $l(ws_i) = l(w) + 1$. Let $h \in A$. We have $\dot{w} x_i(h) \dot{w}^{-1} \in U_A$.

The proof of the following result is similar to that of (a):

(b) Let $z \in W$ and let $i \in I$ be such that $l(s_i z) = l(z) + 1$. Let $h \in A$. Then we have $\dot{z}^{-1} y_i(h) \dot{z} \in U_A^-$.

2.6. Let $(i_1, \dots, i_n) \in \mathcal{X}$. For any $(h_1, h_2, \dots, h_n) \in A^n$ we have

$$x_{i_1}(h_1) \dot{s}_{i_1} x_{i_2}(h_2) \dot{s}_{i_2} \cdots x_{i_n}(h_n) \dot{s}_{i_n} \dot{w}_I^{-1}$$

$$= x_{i_1}(h_1) (\dot{s}_{i_1} x_{i_2}(h_2) \dot{s}_{i_1}^{-1}) \cdots (\dot{s}_{i_1} \dot{s}_{i_2} \cdots \dot{s}_{i_{n-1}} x_{i_n}(h_n) \dot{s}_{i_{n-1}}^{-1} \cdots \dot{s}_{i_2}^{-1} \dot{s}_{i_1}^{-1}) \in U_A.$$

(We use 2.5(a).) From [L2, 4.8(a)] we see that

(a) the map $A^n \rightarrow U_A, (h_1, h_2, \dots, h_n) \mapsto x_{i_1}(h_1) \dot{s}_{i_1} x_{i_2}(h_2) \dot{s}_{i_2} \cdots x_{i_n}(h_n) \dot{s}_{i_n} \dot{w}_I^{-1}$ is a bijection. ■

2.7. Let $w \in W$. Let

$$U_A^w = U_A \cap \dot{w}U_A^- \dot{w}^{-1}, \quad {}^wU_A = U_A \cap \dot{w}U_A \dot{w}^{-1};$$

these are subgroups of U_A . We can find $(i_1, \dots, i_n) \in \mathcal{X}$ such that $s_{i_1}s_{i_2}\dots s_{i_k} = w$ where $k = l(w)$. Let $(h_1, h_2, \dots, h_n) \in A^n$ and let $u \in U_A$ be its image under the map 2.6(a). We have $u = u'u''$ where $u' = r_1r_2\dots r_k$, $u'' = r_{k+1}r_{k+2}\dots r_n$ and

$$r_m = \dot{s}_{i_1}\dot{s}_{i_2}\dots\dot{s}_{i_{m-1}}x_{i_m}(h_m)\dot{s}_{i_{m-1}}^{-1}\dots\dot{s}_{i_2}^{-1}\dot{s}_{i_1}^{-1} \in U_A$$

for $m \in [1, n]$. If $m \in [1, k]$ we have

$$\begin{aligned} \dot{w}^{-1}r_m\dot{w} &= \dot{s}_{i_k}^{-1}\dot{s}_{i_{k-1}}^{-1}\dots\dot{s}_{i_m}^{-1}x_{i_m}(h_m)\dot{s}_{i_m}\dots\dot{s}_{i_{k-1}}\dot{s}_{i_k} \\ &= \dot{s}_{i_k}^{-1}\dot{s}_{i_{k-1}}^{-1}\dots\dot{s}_{i_{m+1}}^{-1}y_{i_m}(-h_m)\dot{s}_{i_{m+1}}\dots\dot{s}_{i_{k-1}}\dot{s}_{i_k} \in U_A^- \end{aligned}$$

(we have used 2.4(c), 2.5(b)). Hence $\dot{w}^{-1}u'\dot{w} \in U_A^-$ and $u' \in U_A^w$. If $m \in [k+1, n]$ we have

$$\dot{w}^{-1}r_m\dot{w} = \dot{s}_{i_{k+1}}\dot{s}_{i_{k+2}}\dots\dot{s}_{i_{m-1}}x_{i_m}(h_m)\dot{s}_{i_{m-1}}^{-1}\dots\dot{s}_{i_{k+2}}^{-1}\dot{s}_{i_{k+1}}^{-1} \in U_A$$

(we have used 2.5(a)). Hence $\dot{w}^{-1}u''\dot{w} \in U_A$ and $u'' \in {}^wU_A$. If we assume that $u \in U_A^w$ then $u'' = u'^{-1}u \in U_A^w$ hence $\dot{w}^{-1}u''\dot{w} \in U_A^-$. We have also $\dot{w}^{-1}u''\dot{w} \in U_A$. Since $U_A \cap U_A^- = \{1\}$ (by 2.3(b)) we have $\dot{w}^{-1}u''\dot{w} = 1$ and $u'' = 1$, $u = u'$. Conversely, if $u = u'$ then, as we have seen, we have $u \in U_A^w$.

If we assume that $u \in {}^wU_A$ then $u' = uu''^{-1} \in {}^wU_A$ hence $\dot{w}^{-1}u'\dot{w} \in U_A$. We have also $\dot{w}^{-1}u'\dot{w} \in U_A^-$. Since $U_A \cap U_A^- = \{1\}$ (by 2.3(b)) we have $\dot{w}^{-1}u'\dot{w} = 1$ and $u' = 1$, $u = u''$. Conversely, if $u = u''$ then, as we have seen, we have $u \in {}^wU_A$. Thus we have the following results:

(a) *the restriction of the map 2.6(a) to A^k (identified with*

$$\{(h_1, h_2, \dots, h_n) \in A^n; h_{k+1} = h_{k+2} = \dots = h_n = 0\}$$

in the obvious way) defines a bijection $A^k \xrightarrow{\sim} U_A^w$;

(b) *the restriction of the map 2.6(a) to A^{n-k} (identified with*

$$\{(h_1, h_2, \dots, h_n) \in A^n; h_1 = h_2 = \dots = h_k = 0\}$$

in the obvious way) defines a bijection $A^{n-k} \xrightarrow{\sim} {}^wU_A$.

Using (a),(b) and 2.6 we see also that

(c) *multiplication defines a bijection $U_A^w \times {}^wU_A \xrightarrow{\sim} U_A$.*

We show:

(d) *multiplication in G_A defines a bijection $(U_A^w \dot{w}) \times U_A \xrightarrow{\sim} U_A \dot{w} U_A$.*

Using (c) we see that $U_A \dot{w} U_A = U_A^w ({}^wU_A) \dot{w} U_A = U_A^w \dot{w} (\dot{w}^{-1} {}^wU_A \dot{w}) U_A \subset U_A^w \dot{w} U_A$.

Thus the map in (d) is surjective. Assume now that $u_1, u_2 \in U_A^w$ and $u'_1, u'_2 \in U_A$ satisfy $u_1 \dot{w} u'_1 = u_2 \dot{w} u'_2$. Then $\dot{w}^{-1} u_2^{-1} u_1 \dot{w} = u'_2 u'_1{}^{-1}$ is both in U_A^- and in U_A hence by 2.3(b) it is 1. Thus $u_1 = u_2$ and $u'_1 = u'_2$. Thus the map in (d) is injective. This proves (d).

Combining (d) with (a) and 2.6(a), we obtain a bijection

$$(e) \quad A^k \times A^n \xrightarrow{\sim} U_A \dot{w} U_A,$$

$$\begin{aligned} ((h_1, \dots, h_k), (h'_1, \dots, h'_n)) &\mapsto (x_{i_1}(h_1) \dot{s}_{i_1} x_{i_2}(h_2) \dot{s}_{i_2} \dots x_{i_k}(h_k) \dot{s}_{i_k} \dot{w}^{-1}) \dot{w} \\ &\quad (x_{i_1}(h'_1) \dot{s}_{i_1} x_{i_2}(h'_2) \dot{s}_{i_2} \dots x_{i_n}(h'_n) \dot{s}_{i_n} \dot{w}_I^{-1}). \end{aligned}$$

We can reformulate (a) as follows.

(f) *If j_1, j_2, \dots, j_k is a sequence in I such that $w = s_{j_1} \dots s_{j_k}$, $l(w) = k$, then the map $A^k \rightarrow U_A^w \dot{w}$,*

$$(h_1, h_2, \dots, h_k) \mapsto x_{i_1}(h_1) \dot{s}_{i_1} x_{i_2}(h_2) \dot{s}_{i_2} \dots x_{i_k}(h_k) \dot{s}_{i_k}$$

is a bijection.

For any sequence $w_* = (w_1, w_2, \dots, w_r)$ in W we set

$$U(w_*) = (U_A^{w_1} \dot{w}_1) \times (U_A^{w_2} \dot{w}_2) \times \dots \times (U_A^{w_r} \dot{w}_r),$$

$$\dot{U}(w_*) = (U_A \dot{w}_1 U_A) \times (U_A \dot{w}_2 U_A) \times \dots \times (U_A \dot{w}_r U_A).$$

We have an obvious inclusion $U(w_*) \subset \dot{U}(w_*)$. Let $\tilde{U}(w_*)$ be the set of orbits of the U_A^{r-1} -action

$$(u_1, u_2, \dots, u_{r-1}) : (g_1, g_2, \dots, g_r) \mapsto (g_1 u_1^{-1}, u_1 g_2 u_2^{-1}, \dots, u_{r-1} g_r)$$

on $\dot{U}(w_*)$. Let $\kappa_{w_*} : \dot{U}(w_*) \rightarrow \tilde{U}(w_*)$ be the obvious surjective map; for $(g_1, \dots, g_r) \in \dot{U}(w_*)$ we set $[g_1, \dots, g_r] = \kappa_{w_*}(g_1, \dots, g_r)$.

The following result is an immediate consequence of (f).

(g) *Let $w \in W$ and let $w_* = (w_1, w_2, \dots, w_r)$ be a sequence in W such that $w = w_1 w_2 \dots w_r$, $l(w) = l(w_1) + l(w_2) + \dots + l(w_r)$. Then multiplication in G_A defines a bijection $\phi_{w_*} : U(w_*) \xrightarrow{\sim} U_A^w \dot{w}$.*

We show:

(h) *Let $x, y \in W$ be such that $l(xy) = l(x) + l(y)$. Let $x_* = (x, y)$. Then multiplication in G_A defines a bijection $\tilde{U}(x_*) \xrightarrow{\sim} U_A \dot{x} \dot{y} U_A$.*

This follows from (d) and (g) since

$$\tilde{U}(x_*) \cong (U_A^x \dot{x}) \times (U_A \dot{y} U_A) = (U_A^x \dot{x}) \times (U_A^y \dot{y} U_A) = U_A^{xy} \dot{x} \dot{y} U_A.$$

Using (h) repeatedly we obtain:

(i) *In the setup of (g), multiplication in G_A defines a bijection $\psi_{w_*} : \tilde{U}(w_*) \xrightarrow{\sim} U_A \dot{w} U_A$. This bijection is compatible with the $U_A \times U_A$ actions*

$$(u, u') : [g_1, g_2, \dots, g_r] \mapsto [u g_1, g_2, \dots, g_r u'^{-1}]$$

on $\tilde{U}(w_)$ and $(u, u') : g \mapsto u g u'^{-1}$ on $U_A \dot{x} U_A$.*

2.8. Let $y_* = (y_1, y_2, \dots, y_t)$ ($t \geq 2$) be a sequence in W . Let $\xi \in \tilde{U}(y_*)$. We show:

(a) *for any $s \in [1, t-1]$ there exists a representative of ξ in*

$$(U_A^{y_1} \dot{y}_1) \times \dots \times (U_A^{y_{s-1}} \dot{y}_{s-1}) \times (U_A \dot{y}_s U_A) \times \dots \times (U_A \dot{y}_t U_A).$$

We argue by induction on s . Let $(g_1, g_2, \dots, g_t) \in \dot{U}(y_*)$ be a representative of ξ . We have $g_1 = u \dot{y}_1 u'$ where $u \in U_A^{y_1}, u' \in U_A$ (see 2.7(d)). Then $(u \dot{y}_1, u' g_2, g_3, \dots, g_t)$ represents ξ and is as in (a) with $s = 1$. Now assume that $s \geq 2$ and that (g_1, g_2, \dots, g_t) is a representative of ξ as in (a) with s replaced by $s-1$. We have $g_{s-1} = u \dot{y}_{s-1} u'$ with $u \in U_A^{y_{s-1}}, u' \in U_A$ (see 2.7(d)). Then

$$(g_1, g_2, \dots, g_{s-2}, u \dot{y}_{s-1}, u' g_s, g_{s+1}, \dots, g_t)$$

is a representative of ξ as in (a). This completes the inductive proof of (a).

We show:

(b) *there exist unique elements $u_1 \in U_A^{y_1}, \dots, u_t \in U_A^{y_t}$ and $u \in U_A$ such that $\xi = [u_1 \dot{y}_1, \dots, u_{t-1} \dot{y}_{t-1}, u_t \dot{y}_t u]$.*

The existence of these elements follows from (a) with $s = t-1$ and 2.7(d). We prove uniqueness. Assume that $u_1, u'_1 \in U_A^{y_1}, \dots, u_t, u'_t \in U_A^{y_t}$ and $u, u' \in U_A$ are such that

$$[u_1 \dot{y}_1, \dots, u_{t-1} \dot{y}_{t-1}, u_t \dot{y}_t u] = [u'_1 \dot{y}_1, \dots, u'_{t-1} \dot{y}_{t-1}, u'_t \dot{y}_t u'] = \xi.$$

Then there exist v_1, v_2, \dots, v_{t-1} in U_A such that

$$\begin{aligned} u'_1 \dot{y}_1 &= u_1 \dot{y}_1 v_1, u'_2 \dot{y}_2 = v_1^{-1} u_2 \dot{y}_2 v_2, \dots, u'_{t-1} \dot{y}_{t-1} = v_{t-2}^{-1} u_{t-1} \dot{y}_{t-1} v_{t-1}, \\ u'_t \dot{y}_t u' &= v_{t-1}^{-1} u_t \dot{y}_t u. \end{aligned}$$

The first of these equations implies (using 2.7(d)) that $v_1 = 1$ and $u'_1 = u_1$. Then the second equation becomes $u'_2 \dot{y}_2 = u_2 \dot{y}_2 v_2$; using again 2.7(d) we deduce that $v_2 = 1$ and $u'_2 = u_2$. Continuing in this way we get $v_1 = \dots = v_{t-1} = 1$ and $u_i = u'_i$ for $i \in [1, t-1]$. We then have $u'_t \dot{y}_t u' = u_t \dot{y}_t u$. Using 2.7(d) we deduce $u'_t = u_t, u' = u$. This proves (b).

We now define $\zeta : \tilde{U}(y_*) \rightarrow U_A$ by $\zeta(\xi) = u$ where u is as in (b).

2.9. Let \hat{W} be the braid monoid attached to W and let $w \mapsto \hat{w}$ be the canonical map $W \rightarrow \hat{W}$. Let $x_* = (x_1, \dots, x_s), y_* = (y_1, \dots, y_t)$ be two sequences in W such that $\hat{x}_1 \dots \hat{x}_s = \hat{y}_1 \dots \hat{y}_t$ in \hat{W} . We show:

(a) *there exist bijections $H : U(x_*) \xrightarrow{\sim} U(y_*)$, $\tilde{H} : \tilde{U}(x_*) \xrightarrow{\sim} \tilde{U}(y_*)$ such that $\kappa_{y_*} H = \tilde{H} \kappa_{x_*}$ and such that \tilde{H} is compatible with the $U_A \times U_A$ -actions (as in 2.7(i)).*

Applying 2.7(g) and 2.7(i) with w replaced by x_a (resp. y_b) and w_1, w_2, \dots, w_r

replaced by a sequence of simple reflections whose product is a reduced expression of x_a (resp. y_b) we see that the general case is reduced to the case where $l(x_1) = \dots = l(x_s) = l(y_1) = \dots = l(y_t) = 1$; in this case we must have $s = t$. Since $\hat{x}_1 \dots \hat{x}_s = \hat{y}_1 \dots \hat{y}_s$ we can find a sequence $\mathbf{s}_*^1, \mathbf{s}_*^2, \dots, \mathbf{s}_*^m$ where each \mathbf{s}_*^p is a sequence in S , $\mathbf{s}_*^1 = x_*$, $\mathbf{s}_*^m = y_*$ and for any $r \in [1, m-1]$, \mathbf{s}_*^{r+1} is related to \mathbf{s}_*^r as follows:

(b) for some $i \neq j$ in I and some e such that $[e+1, e+u] \subset [1, s]$ ($n_{ij} = u$) we have

$$\begin{aligned} \mathbf{s}_c^r &= \mathbf{s}_c^{r+1} \text{ if } c \in [1, s] - [e+1, e+u], \\ (\mathbf{s}_{e+1}^r, \mathbf{s}_{e+2}^r, \dots, \mathbf{s}_{e+u}^r) &= s_* := (s_i, s_j, s_i, \dots), \\ (\mathbf{s}_{e+1}^{r+1}, \mathbf{s}_{e+2}^{r+1}, \dots, \mathbf{s}_{e+u}^{r+1}) &= s'_* := (s_j, s_i, s_j, \dots). \end{aligned}$$

Note that each pair $\mathbf{s}_*^r, \mathbf{s}_*^{r+1}$ satisfies the same assumptions as the pair x_*, y_* . If (a) can be proved for each $\mathbf{s}^r, \mathbf{s}^{r+1}$ then (a) would follow also for x_*, y_* (by taking appropriate compositions of maps like H or maps like \tilde{H}). It is then enough to prove (a) assuming that $x_* = \mathbf{s}_*^r$, $y_* = \mathbf{s}_*^{r+1}$ are as in (b). Let $w = s_i s_j s_i \dots = s_j s_i s_j \dots$ (u factors). Let

$$x'_* = (x_1, \dots, x_e) = (y_1, \dots, y_e), \quad x''_* = (x_{e+u+1}, \dots, x_s) = (x_{e+u+1}, \dots, x_s).$$

Then

$$U(x_*) = U(x'_*) \times U(s_*) \times U(x''_*), \quad U(y_*) = U(x'_*) \times U(s'_*) \times U(x''_*).$$

Now $\tilde{U}(x_*)$ (resp. $\tilde{U}(y_*)$) is the set of orbits for the action of a subgroup of $U_A \times U_A$ on $\tilde{U}(x'_*) \times \tilde{U}(s_*) \times \tilde{U}(x''_*)$ (resp. $\tilde{U}(x'_*) \times \tilde{U}(s'_*) \times \tilde{U}(x''_*)$) given by

$$(u, u') : (\xi, \xi', \xi'') \mapsto ((1, u)\xi, (u, u')\xi', (u', 1)\xi'')$$

(the subgroup is $U_A \times U_A$ if x'_*, x''_* are nonempty, is $\{1\} \times U_A$ if x'_* is empty and x''_* is nonempty, is $U_A \times \{1\}$ if x'_* is nonempty and x''_* is empty, is $\{1\} \times \{1\}$ if x'_* and x''_* are empty.) Let \mathcal{T} be the set of orbits of the same subgroup of $U_A \times U_A$ on $\tilde{U}(x'_*) \times (U_A \dot{w} U_A) \times \tilde{U}(x''_*)$ for the action given by the same formulas as above. We have a diagram

$$U(x_*) \xrightarrow{h} U(x'_*) \times (U_A^w \dot{w}) \times U(x''_*) \xleftarrow{h'} U(y_*)$$

where

$$h(\xi, \xi', \xi'') = (\xi, \phi_{s_*}(\xi'), \xi''), \quad h'(\xi, \xi', \xi'') = (\xi, \phi_{s'_*}(\xi'), \xi'')$$

(with $\phi_{s_*}, \phi_{s'_*}$ as in 2.7(g). Note that h, h' are bijections. We set $H = h'^{-1}h$. We have a diagram

$$\tilde{U}(x_*) \xrightarrow{\tilde{h}} \mathcal{T} \xleftarrow{\tilde{h}'} \tilde{U}(y_*)$$

where

$$\tilde{h}(\xi, \xi', \xi'') = (\xi, \psi_{s_*}(\xi'), \xi''), \quad \tilde{h}'(\xi, \xi', \xi'') = (\xi, \psi_{s'_*}(\xi'), \xi'')$$

(with $\psi_{s_*}, \psi_{s'_*}$ as in 2.7(i)). Note that \tilde{h}, \tilde{h}' are bijections (they are well defined by 2.7(i)). We set $\tilde{H} = (\tilde{h}')^{-1}\tilde{h}$. It is clear that H, \tilde{H} satisfy the requirements of (a). This proves (a).

2.10. Let δ be an automorphism of \mathcal{R} that is, a triple consisting of automorphisms $\delta : Y \rightarrow Y$, $\delta : X \rightarrow X$ and a bijection $\delta : I \rightarrow I$ such that $\langle \delta(y), \delta(x) \rangle = \langle y, x \rangle$ for $y \in Y, x \in X$, δ is compatible with the imbeddings $I \rightarrow Y$, $I \rightarrow X$ and $\delta(i) \cdot \delta(j) = i \cdot j$ for $i, j \in I$. There is a unique group automorphism of W , $w \mapsto \delta(w)$ such that $\delta(s_i) = s_{\delta(i)}$ for all $i \in I$. There is a unique algebra automorphism (preserving 1) of \mathbf{f}_A , $x \mapsto \delta(x)$, such that $\delta(\theta_i^{(c)}) = \theta_{\delta(i)}^{(c)}$ for all $i \in I, c \in \mathbf{N}$. We have $\delta(\mathbf{B}) = \mathbf{B}$. There is a unique algebra automorphism of $\dot{\mathbf{U}}_A$, $u \mapsto \delta(u)$, such that $\delta(b^{-1}\zeta b'^+) = \delta(b)^{-1}\delta(\zeta)\delta(b')^+$ for all $b \in \mathbf{B}, b' \in \mathbf{B}, \zeta \in X$. We have $\delta(\dot{\mathbf{B}}) = \dot{\mathbf{B}}$. There is a unique algebra automorphism (preserving 1) of $\hat{\mathbf{U}}_A$, $u \mapsto \delta(u)$ such that $\delta(\sum_{a \in \dot{\mathbf{B}}} n_a a) = \sum_{a \in \dot{\mathbf{B}}} n_a \delta(a)$ for all functions $\dot{\mathbf{B}} \rightarrow A$, $a \mapsto n_a$. This automorphism restricts to a group automorphism $G_A \rightarrow G_A$ denoted again by δ and to an automorphism of U_A . For any $i \in I, h \in H$ we have $\delta(x_i(h)) = x_{\delta(i)}(h)$, $\delta(\dot{s}_i) = \dot{s}_{\delta(i)}$.

2.11. Let $A \in \mathcal{C}, A' \in \mathcal{C}$ and let $\chi : A \rightarrow A'$ be a homomorphism of rings preserving 1. There is a unique ring homomorphism (preserving 1) $\hat{\mathbf{U}}_A \rightarrow \hat{\mathbf{U}}_{A'}$, $u \mapsto \chi(u)$ such that $\chi(\sum_{a \in \dot{\mathbf{B}}} n_a a) = \sum_{a \in \dot{\mathbf{B}}} \chi(n_a) a$ for all functions $\dot{\mathbf{B}} \rightarrow A$, $a \mapsto n_a$. This restricts to a group homomorphism $G_A \rightarrow G_{A'}$ denoted again by χ and to a group homomorphism $U_A \rightarrow U_{A'}$. For any $i \in I, h \in A$ we have $\chi(x_i(h)) = x_i(\chi(h))$, $\chi(\dot{s}_i) = \dot{s}_i$.

3. THE MAIN RESULTS

3.1. In this section $A \in \mathcal{C}$ is fixed unless otherwise specified. We also fix an automorphism δ of \mathcal{R} as in 2.10 and a ring automorphism χ of A preserving 1. There are induced group automorphisms of G_A denoted again by δ, χ (see 2.10, 2.11). These automorphisms commute; we set $\pi = \delta\chi = \chi\delta : G_A \rightarrow G_A$; note that π maps U_A onto itself.

Two elements w, w' of W are said to be δ -conjugate if $w' = y^{-1}w\delta(y)$ for some $y \in W$. The relation of δ -conjugacy is an equivalence relation on W ; the equivalence classes are said to be δ -conjugacy classes. A δ -conjugacy class C in W (or an element of it) is said to be δ -elliptic if $C \cap W_J = \emptyset$ for any $J \subsetneq I, \delta(J) = J$. Let C be a δ -elliptic δ -conjugacy class in W . Let C_{min} be the set of elements of minimal length of C .

For any $w \in W$ we define a map

$$\Xi_A^w : U_A \times (U_A^w \dot{w}) \rightarrow U_A \dot{w} U_A$$

by $(u, z) \mapsto uz\pi(u)^{-1}$.

3.2. In this subsection we assume that $x, y \in W$ are such that $l(x\delta(y)) = l(x) + l(y) = l(yx)$. We show:

(*) $\Xi_A^{x\delta(y)}$ is injective if and only if Ξ_A^{yx} is injective.

Let $x_* = (x, \delta(y)), x'_* = (y, x)$. In the following proof we write $U, U^x, U^y, U^{\delta(y)}$

instead of $U_A, U_A^x, U_A^y, U_A^{\delta(y)}$. Now $\Xi_A^{x\delta(y)}$ can be identified with

$$U \times (U^x \dot{x}) \times (U^{\delta(y)} \delta(\dot{y})) \rightarrow \tilde{U}(x_*), \quad (u, z, z') \mapsto [uz, z' \pi(u)^{-1}]$$

and Ξ_A^{yx} can be identified with

$$U \times (U^y \dot{y}) \times (U^x \dot{x}) \rightarrow \tilde{U}(x'_*), \quad (u, z', z) \mapsto [uz', z \pi(u)^{-1}].$$

(We use 2.7(g), 2.7(i).) The condition that $\Xi_A^{x\delta(y)}$ is injective can be stated as follows:

(a) if $u \in U, z_1, z_2 \in U^x \dot{x}, z_3, z_4 \in U^{\delta(y)} \delta(\dot{y})$ satisfy $uz_1 v^{-1} = z_2, v z_3 \pi(u)^{-1} = z_4$ for some $v \in U$ then $u = 1$.

The condition that Ξ_A^{yx} is injective can be stated as follows:

(b) if $u' \in U, z'_1, z'_2 \in U^x \dot{x}, z'_3, z'_4 \in U^y \dot{y}$ satisfy $u' z'_3 v'^{-1} = z'_4, v' z'_1 \pi(u')^{-1} = z'_2$ for some $v' \in U$ then $u' = 1$.

Assume that (a) holds and that the hypothesis of (b) holds. We have $v' z'_1 \pi(u')^{-1} = z'_2, \pi(u') \pi(z'_3) \pi(v')^{-1} = \pi(z'_4)$ and $\pi(z'_3), \pi(z'_4) \in U^{\delta(y)} \delta(\dot{y})$. Applying (a) with $v = \pi(u'), u = v'$ we obtain $v' = 1$. Then $\pi(u') = z'_2{}^{-1} z'_1 \in \dot{x}^{-1} U^x \dot{x} \subset U^-$. But $U \cap U^- = \{1\}$ (see 2.3(b)) hence $u' = 1$. Thus the conclusion of (b) holds.

Next we assume that (b) holds and that the hypothesis of (a) holds. We have $\pi^{-1}(v) \pi^{-1}(z_3) u^{-1} = \pi^{-1}(z_4)$ and $\pi^{-1}(z_3), \pi^{-1}(z_4) \in U^y \dot{y}$. Applying (b) with $v' = u, u' = \pi^{-1}(v)$ we obtain $v = 1$. Then $\pi(u) = z_4{}^{-1} z_3 \in \dot{y}^{-1} U^y \dot{y} \subset U^-$. But $U \cap U^- = \{1\}$ hence $u = 1$. Thus the conclusion of (a) holds. We see that (a) holds if and only if (b) holds. This proves (*).

3.3. In this subsection we assume that $x, y \in W$ are such that $l(yx) = l(x) + l(y)$ and we write U, U^x, U^y, U^{yx} instead of $U_A, U_A^x, U_A^y, U_A^{yx}$. We have a commutative diagram

$$\begin{array}{ccc} U \times U \times (U^y \dot{y}) \times (U^x \dot{x}) & \xrightarrow{\Theta_A} & (U \dot{y} U) \times (U \dot{x} U) \\ m \downarrow & & m' \downarrow \\ U \times (U^{yx} \dot{y} \dot{x}) & \xrightarrow{\Xi_A^{yx}} & U \dot{y} \dot{x} U \end{array}$$

where

$$\Theta_A(u, u', g, g') = (ugu'^{-1}, u'g'\pi(u)^{-1}), \quad m(u, u', g, g') = (u, gg'), \quad m'(g, g') = gg'. \blacksquare$$

We show

(a) the sets

$$S' = \{\Xi' : U \dot{y} \dot{x} U \rightarrow U \times (U^{yx} \dot{y} \dot{x}); \Xi_A^{yx} \Xi' = 1\},$$

$$S'' = \{Z : (U \dot{y} U) \times (U \dot{x} U) \rightarrow U \times U \times (U^y \dot{y}) \times (U^x \dot{x}); \Theta_A Z = 1\}$$

are in natural bijection.

Let $\Xi' \in S'$. We define Z as follows. Let $(z, z') \in (U\dot{y}U) \times (U\dot{x}U)$. We have $\Xi'(zz') = (u, gg')$ where $u \in U$, $g \in U^y\dot{y}$, $g' \in U^x\dot{x}$ are uniquely defined. Since $ug \in U\dot{y}U$ we can write uniquely $ug = z_0u'$ where $z_0 \in U^y\dot{y}$, $u' \in U$ (see 2.7(d)). We set $Z(z, z') = (u, u', g, g')$. This defines the map Z . We have

$$zz' = \Xi_A^{yx}(\Xi'(zz')) = ugg'\pi(u)^{-1}.$$

The equality $zz' = (ug)(g'\pi(u)^{-1})$ and 2.7(h) imply that $ugv^{-1} = z, vg'\pi(u)^{-1} = z'$ for a well defined $v \in U$. We have $ug = zv = z_0u'$. From $zv = z_0u'$ and 2.7(d) we see that $u' = v$. Thus $ugu'^{-1} = z, u'g'\pi(u)^{-1} = z'$. Hence

$$\Theta_A(Z(z, z')) = \Theta_A(u, u', g, g') = (ugu'^{-1}, u'g'\pi(u)^{-1}) = (z, z').$$

Thus $\Theta_A Z = 1$ and $Z \in S''$.

Conversely, let $Z \in S''$. We define Ξ' as follows. Let $h \in U\dot{y}\dot{x}U$. Using 2.7(h) and 2.7(d) we can write uniquely $h = zz'$ where $z \in U^y\dot{y}$, $z' \in U^x\dot{x}U$. We set $\Xi'(h) = m(Z(z, z'))$. We have

$$\Xi_A^{yx}(\Xi'(h)) = \Xi_A^{yx}(m(Z(z, z'))) = m'(\Theta_A(Z(z, z'))) = m'(z, z') = zz' = h.$$

Thus $\Xi_A^{yx}\Xi' = 1$ and $\Xi' \in S'$.

It is easy to check that the maps $S' \rightarrow S''$, $S'' \rightarrow S'$ defined above are inverse to each other. This proves (a).

3.4. In this subsection we assume that $x, y \in W$ are such that

$$l(x\delta(y)) = l(x) + l(y) = l(yx)$$

and we write $U, U^x, U^y, U^{\delta(y)}, U^{yx}, U^{x\delta(y)}$ instead of $U_A, U_A^x, U_A^y, U_A^{\delta(y)}, U_A^{yx}, U_A^{x\delta(y)}$. We show
(a) the sets

$$S'_1 = \{\Xi'_1 : U\dot{y}\dot{x}U \rightarrow U \times (U^{yx}\dot{y}\dot{x}); \Xi_A^{yx}\Xi'_1 = 1\},$$

$$S'_2 = \{\Xi'_2 : U\dot{x}\delta(\dot{y})U \rightarrow U \times (U^{x\delta(y)}\dot{x}\delta(\dot{y})); \Xi_A^{yx}\Xi'_2 = 1\}$$

are in natural bijection.

Define $\Theta : U \times U \times (U^y\dot{y}) \times (U^x\dot{x}) \rightarrow (U\dot{y}U) \times (U\dot{x}U)$ and

$$\Theta' : U \times U \times (U^x\dot{x}) \times (U^{\delta(y)}\delta(\dot{y})) \rightarrow (U\dot{x}U) \times (U\delta(\dot{y})U)$$

by $(u, u', g, g') \mapsto (ugu'^{-1}, u'g'\pi(u)^{-1})$. In view of 3.3 applied to x, y and also to $\delta(y), x$ we see that to prove (a), it is enough to show:

(b) the sets

$$S''_1 = \{Z : (U\dot{y}U) \times (U\dot{x}U) \rightarrow U \times U \times (U^y\dot{y}) \times (U^x\dot{x}); \Theta Z = 1\},$$

$$S_2'' = \{Z' : (U\dot{x}U) \times (U\delta(\dot{y})U) \rightarrow U \times U \times (U^x\dot{x}) \times (U^{\delta(y)}\delta(\dot{y})); \Theta'Z' = 1\}$$

are in natural bijection.

here

Now (b) follows from the commutative diagram

$$\begin{array}{ccc} U \times U \times (U^y\dot{y}) \times (U^x\dot{x}) & \xrightarrow{\Theta} & (U\dot{y}U) \times (U\dot{x}U) \\ \iota \downarrow & & \iota' \downarrow \\ U \times U \times (U^x\dot{x}) \times (U^{\delta(y)}\delta(\dot{y})) & \xrightarrow{\Theta'} & (U\dot{x}U) \times (U\delta(\dot{y})U) \end{array}$$

where $\iota(u, u', g, g') = (u', \pi(u), g', \pi(g))$ and $\iota'(g, g') = (g', \pi(g))$ are bijections. This proves (a).

3.5. We show:

(a) if Ξ_A^w is injective for some $w \in C_{min}$ then Ξ_A^w is injective for any $w \in C_{min}$; For any w, w' in C_{min} there exists a sequence $w = w_1, w_2, \dots, w_r = w'$ in C_{min} such that for any $h \in [1, r-1]$ we have either $w_h = x\delta(y)$, $w_{h+1} = yx$ for some x, y as in 3.2 or $w_{h+1} = x\delta(y)$, $w_h = yx$ for some x, y as in 3.2 (See [GP], [GKP], [He].) Now (a) follows by applying 3.2(*) several times.

Now assume that for some $w' \in C_{min}$, we are given $\Xi' : U_A \dot{w}' U_A \rightarrow U_A \times (U_A^{w'} \dot{w}')$ such that $\Xi_A^{w'} \Xi' = 1$. Let $w \in C_{min}$. We show how to construct a map $\Xi'' : U_A \dot{w} U_A \rightarrow U_A \times (U_A^w \dot{w})$ such that $\Xi_A^w \Xi'' = 1$. We choose a sequence $w = w_1, w_2, \dots, w_r = w'$ in C_{min} as in the proof of (a). We define a sequence of maps $\Xi'_i : U_A \dot{w}_i U_A \rightarrow U_A \times (U_A^{w_i} \dot{w}_i)$ ($i \in [1, r]$) such that $\Xi_A^{w_i} \Xi'_i = 1$ by induction on i as follows. We set $\Xi'_1 = \Xi'$. Assuming that Ξ'_i is defined for some $i \in [1, r-1]$ we define Ξ'_{i+1} so that Ξ'_i, Ξ'_{i+1} correspond to each other under a bijection as in 3.4. Then the map $\Xi'' := \Xi'_r$ satisfies our requirement. In particular, we see that:

(b) if Ξ_A^w is surjective for some $w \in C_{min}$ then Ξ_A^w is surjective for any $w \in C_{min}$.

Theorem 3.6. Recall that $A \in \mathcal{C}$. Let C be a δ -elliptic δ -conjugacy class in W and let $w \in C_{min}$. Then:

- (i) Ξ_A^w is injective;
- (ii) if $\chi = 1$, then Ξ_A^w is bijective;
- (iii) if A is a field, χ has finite order m and the fixed point field A^χ is perfect, then Ξ_A^w is bijective;
- (iv) if A is an algebraic closure of a finite field F_q and $\chi(x) = x^q$ for all $x \in A$ then Ξ_A^w is bijective;
- (v) if A is finite and χ is arbitrary then Ξ_A^w is bijective.

The proof will occupy 3.7-3.10. If $W = \{1\}$ the result is trivial. Hence we can assume that $W \neq \{1\}$.

3.7. Let e be the order of any element of C . According to [GM], [GKP], [He], we can find an element $w' \in C_{min}$ such that $\widehat{w'\delta(w')\dots\delta^{e-1}(w')} = \hat{y}_1\hat{y}_2\dots\hat{y}_t$ (in the braid monoid \hat{W}) where $y_* = (y_1, y_2, \dots, y_t)$ is a sequence in W such that $y_1 = w_I$. Since $W \neq \{1\}$ we have $t \geq 2$. From now until the end of 3.9 we assume that $w = w'$. Let $x_* = (w, \delta(w), \dots, \delta^{e-1}(w))$ and let $H : U(x_*) \xrightarrow{\sim} U(y_*)$, $\tilde{H} : U(x_*) \xrightarrow{\sim} U(y_*)$ be as in 2.9(a). Let $z \in U_A \dot{w} U_A$. Then $[z, \pi(z), \dots, \pi^{e-1}(z)] \in \tilde{U}(x_*)$ hence $\tilde{H}[z, \pi(z), \dots, \pi^{e-1}(z)] \in \tilde{U}(y_*)$. We set $u = \zeta(\tilde{H}[z, \pi(z), \dots, \pi^{e-1}(z)]) \in U_A$ where $\zeta : U(y_*) \rightarrow U$ is as in 2.8. We can write uniquely $\pi^{-e}(u)z\pi^{-e+1}(u)^{-1} = z'u'$ where $z' \in U_A^w \dot{w}$, $u' \in U_A$ (see 2.7(d)). We set $\Xi'_A(z) = (\pi^{-e}(u)^{-1}, z') \in U_A \times (U_A^w \dot{w})$. Thus we have a map

$$\Xi'_A : U_A \dot{w} U_A \rightarrow U_A \times (U_A^w \dot{w}).$$

We show:

(a) $\Xi'_A \Xi_A^w$ is the identity map of $U_A \times (U_A^w \dot{w})$ into itself.

Let $(u_1, z_1) \in U_A \times (U_A^w \dot{w})$. Let $z = u_1 z_1 \pi(u_1)^{-1}$. We must show that $\Xi'_A(z) = (u_1, z_1)$. Let $\xi = \tilde{H}[z, \pi(z), \dots, \pi^{e-1}(z)] \in \tilde{U}(y_*)$. By 2.8(b) we have $\xi = (1, u^{-1})[h_1, \dots, h_t]$ with $(h_1, \dots, h_t) \in U(y_*)$, $u \in U_A$ uniquely determined; moreover from the definitions we have $u = \zeta(\xi)$. We have

$$\begin{aligned} & [z, \pi(z), \dots, \pi^{e-1}(z)] \\ &= [u_1 z_1 \pi(u_1)^{-1}, \pi(u_1) \pi(z_1) \pi^2(u_1)^{-1}, \dots, \pi^{e-1}(u_1) \pi^{e-1}(z_1) \pi^e(u_1)^{-1}] \\ &= [u_1 z_1, \pi(z_1), \dots, \pi^{e-2}(z_1), \pi^{e-1}(z_1) \pi^e(u_1)^{-1}] \\ &= (u_1, \pi^e(u_1)) [z_1, \pi(z_1), \dots, \pi^{e-2}(z_1), \pi^{e-1}(z_1)] \in \tilde{U}(x_*), \end{aligned}$$

hence

$$\begin{aligned} \xi &= \tilde{H}((u_1, \pi^e(u_1)) [z_1, \pi(z_1), \dots, \pi^{e-2}(z_1), \pi^{e-1}(z_1)]) \\ &= (u_1, \pi^e(u_1)) \tilde{H}([z_1, \pi(z_1), \dots, \pi^{e-2}(z_1), \pi^{e-1}(z_1)]) \\ &= (u_1, \pi^e(u_1)) [a_1, \dots, a_t] \end{aligned}$$

where $(a_1, \dots, a_t) = H(z_1, \pi(z_1), \dots, \pi^{e-2}(z_1), \pi^{e-1}(z_1)) \in U(y_*)$. Thus we have

$$(1, u^{-1}) [h_1, \dots, h_t] = (u_1, \pi^e(u_1)) [a_1, \dots, a_t] \in \tilde{U}(y_*)$$

and

$$[h_1, \dots, h_t] = [u_1 a_1, a_2, \dots, a_{t-1}, a_t \pi^e(u_1)^{-1} u^{-1}] \in \tilde{U}(y_*).$$

Note that $h_i \in U_A^{y_i} \dot{y}_i$ for $i \in [1, t]$, $a_i \in U_A^{y_i} \dot{y}_i$ for $i \in [2, t]$ and $u_1 a_1 \in U_A^{y_1} \dot{y}_1$ (we use that $y_1 = w_I$, $U_A^{w_I} = U_A$). Using the uniqueness part of 2.8(b) we deduce that $\pi^e(u_1)^{-1} u^{-1} = 1$, hence $\pi^{-e}(u) = u_1^{-1}$. Then we have

$$\pi^{-e}(u) z \pi^{-e+1}(u)^{-1} = u_1^{-1} z \pi(u_1) = z_1 \in U_A^w \dot{w}.$$

Using the definition we have $\Xi'_A(z) = (\pi^{-e}(u)^{-1}, z_1) = (u_1, z_1)$. This proves (a).

3.8. If w is as in 3.7 then from 3.7(a) we see that Ξ_A^w is injective. This proves 3.6(i) for this w .

3.9. Let w be as in 3.7. We set $N = l(w) + l(w_I)$. We identify $A^N = U_A \times (U_A^w \dot{w})$ as in 2.6(a), 2.7(e) and $A^N = U_A \dot{w} U_A$ as in 2.7(f). Then Ξ_A^w and Ξ'_A become maps $f_A : A^N \rightarrow A^N$, $f'_A : A^N \rightarrow A^N$ such that $f'_A f_A = 1$. Assuming that $\chi = 1$ and δ is fixed, we see from the definitions that $(f_A)_{A \in \mathcal{C}}$, $(f'_A)_{A \in \mathcal{C}}$ are polynomial families, see 1.1. (Note that the definition of f'_A involves the isomorphisms H, \tilde{H} in 2.8(a). By the proof of 2.8(a) these isomorphisms can be regarded as polynomial families when A varies.) We can now apply 1.3 and we see that f_A is bijective for any A . This proves 3.6(ii) for our w .

Now assume that A, χ, m, A^χ are as in 3.6(iii). Let A_0 be an algebraic closure of $A_1 := A^\chi$. Let $A_2 = A \otimes_{A_1} A_0 \in \mathcal{C}$. Now $f_A : A^N \rightarrow A^N$ is not given by polynomials with coefficients in A ; however, A is an A_1 -vector space of dimension m and f_A can be viewed as a map $A_1^{Nm} \rightarrow A_1^{Nm}$ given by polynomials with coefficients in A_1 . The same polynomials describe the map $f_{A_2} : A_2^N \rightarrow A_2^N$ viewed as a map $A_0^{Nm} \rightarrow A_0^{Nm}$. This last map is injective by 3.8 (applied to A_2) and then it is automatically bijective by [BR] (see also [Ax], [G1, 10.4.11]) applied to the affine space A_0^{Nm} over A_0 . Thus f_{A_2} is bijective. Let $\xi \in A^N$. Then $\xi' := f_{A_2}^{-1}(\xi) \in A_2^N$ is well defined. The Galois group of A_0 over A_1 acts on A_2 (via the action on the second factor) hence on A_2^N . This action is compatible with f_{A_2} and it fixes ξ hence it fixes ξ' . Since A_1 is perfect it follows that $\xi' \in (A \otimes_{A_1} A_1)^N = A^N$. We have $f_A(\xi') = \xi$. Thus f_A is surjective, hence bijective. This proves 3.6(iii) for our w .

Now assume that A, χ are as in 3.6(iv). In this case f_A can be viewed as a map $A^N \rightarrow A^N$ given by polynomials with coefficients in A . This map is injective by 3.8 and then it is automatically bijective by [BR] (see also [Ax], [G1, 10.4.11]) applied to the affine space A^N . Thus f_A is bijective. This proves 3.6(iv) for our w .

Finally assume that A is finite. Then A^N is finite. Since f_A is injective, it is automatically bijective. This proves 3.6(v) for our w .

3.10. Now let w be any element of C_{min} .

Since $\Xi_A^{w'}$ is injective for w' as in 3.7 (see 3.8) we see using 3.5(a) that Ξ_A^w is injective. This proves 3.6(i) for our w .

Now assume that we are in the setup of 3.6(ii),(iii),(iv) or (v). Since $\Xi_A^{w'}$ is bijective for w' as in 3.7 (see 3.9) we see using 3.5(a),(b) that Ξ_A^w is bijective. This completes the proof of Theorem 3.6.

3.11. Recall that $A \in \mathcal{C}$. In this subsection we assume that $\chi = 1$. Let $w \in C_{min}$ (C as in 3.6). We identify $A^N = U_A \times (U_A^w \dot{w})$ (with $N = l(w) + l(w_I)$) as in 2.6(a), 2.7(e) and $A^N = U_A \dot{w} U_A$ as in 2.7(f). Then Ξ_A^w becomes a map $A^N \rightarrow A^N$. From the definitions we see that $(\Xi_{A'}^w)_{A' \in \mathcal{C}}$ is a polynomial family. (Here δ is fixed and $\chi = 1$ for any A' .) Hence $(\Xi_A^w)^* : A[X_1, \dots, X_N] \rightarrow A[X_1, \dots, X_N]$ is defined for any $A \in \mathcal{C}$. We have the following result.

Theorem 3.12. *In the setup of 3.11, $(\Xi_A^w)^*$ is an isomorphism. In particular, if A is an algebraically closed field, then $\Xi_A^w : A^N \rightarrow A^N$ is an isomorphism of algebraic varieties.*

Let $w' \in C_{min}$ be as in 3.7. As we have seen in 3.9, there exists a polynomial family $\Xi'_A : A^N \rightarrow A^N$ ($A \in \mathcal{C}$) such that $\Xi'_A \Xi_A^{w'} = 1$ for any A . Since $\Xi_A^{w'}$ is bijective by 3.6(ii) we must also have $\Xi_A^{w'} \Xi'_A = 1$. Now the method in the paragraph preceding 3.5(b) yields for any A an explicit map $\Xi''_A : A^N \rightarrow A^N$ such that $\Xi_A^w \Xi''_A = 1$. Moreover from the definitions we see that $(\Xi''_A)_{A \in \mathcal{C}}$ is a polynomial family. It follows that $(\Xi''_A)^*$ is defined and $(\Xi''_A)^*(\Xi_A^w)^* = 1$. Since Ξ_A^w is a bijection (see 3.6(ii)), we see that $\Xi_A^w \Xi''_A = 1$ implies $\Xi''_A \Xi_A^w = 1$ hence $(\Xi_A^w)^*(\Xi''_A)^* = 1$. This, together with $(\Xi''_A)^*(\Xi_A^w)^* = 1$ shows that $(\Xi_A^w)^*$ is an isomorphism. This completes the proof of the theorem. Note that the second assertion of the theorem can alternatively be proved using 1.2(ii) and the fact that Ξ_A^w is injective (see 3.6(i)).

3.13. In this subsection we assume that A, χ are as in 3.6(iv). Let $w \in C_{min}$ (C as in 3.6). We identify $A^N = U_A \times (U_A^w \dot{w})$ (with $N = l(w) + l(w_I)$) as in 2.6(a), 2.7(e) and $A^N = U_A \dot{w} U_A$ as in 2.7(f). Then Ξ_A^w becomes a map $A^N \rightarrow A^N$. It is in fact a morphism of algebraic varieties. Exactly as in 3.12 we define a map $\Xi''_A : A^N \rightarrow A^N$ such that $\Xi_A^w \Xi''_A = 1$. But this time Ξ''_A is not a morphism of algebraic varieties but only a quasi-morphism (see [L5, 1.1]). (This is because the definition of Ξ''_A involves $\chi^{-1} : A \rightarrow A$ which is a quasi-morphism but not a morphism.) Since Ξ_A^w is a bijection (see 3.6(iv)) we deduce that we have also $\Xi''_A \Xi_A^w = 1$. Thus we have the following result:

(a) $\Xi_A^w : A^N \rightarrow A^N$ is a bijective morphism whose inverse is a quasi-morphism.

3.14. Let C be as in 3.6 and let $w \in C_{min}$. Let $A \in \mathcal{C}$ and let δ, χ, π be as in 2.1. We define

(a) $\alpha_A^w : \delta^{-1}(w)^{-1} U_A \times U_A^w \rightarrow U_A$ by $(u', u'') \mapsto u' u'' \dot{w} \pi(u')^{-1} \dot{w}^{-1}$.

We show:

(b) α_A^w is injective.

Assume that

$(u', u'') \in \delta^{-1}(w)^{-1} U_A \times U_A^w$, $(u'_1, u''_1) \in \delta^{-1}(w)^{-1} U_A \times U_A^w$ and $u' u'' \dot{w} \pi(u')^{-1} \dot{w}^{-1} = u'_1 u''_1 \dot{w} \pi(u'_1)^{-1} \dot{w}^{-1}$. Then $\Xi_A^w(u', u'' \dot{w}) = \Xi_A^w(u'_1, u''_1 \dot{w})$. Using 3.6(i) we deduce $(u', u'' \dot{w}) = (u'_1, u''_1 \dot{w})$ hence $u' = u'_1$, $u'' = u''_1$ and (a) follows.

We show:

(c) If χ, A are as in 3.6(ii), (iii), (iv) or (v), then α_A^w is bijective.

Let $u \in U_A$. By 3.6, we have $u \dot{w} = u' u'' \dot{w} \pi(u')^{-1}$ for some $u' \in U_A, u'' \in U_A^w$. By 2.7(c) we can write $u'^{-1} u = u_1 u_2$ where $u_1 \in U_A^w, u_2 \in {}^w U_A$. Then $u_1 \dot{w} (\dot{w}^{-1} u_2 \dot{w}) = u'' \dot{w} \pi(u')^{-1}$. Using 2.7(d) we deduce that $\pi(u')^{-1} = \dot{w}^{-1} u_2 \dot{w} \in {}^{w^{-1}} U_A$. Thus $u' \in \delta^{-1}(w)^{-1} U_A$ so that α_A^w is surjective. Together with (b) this implies that α_A^w is bijective.

We note:

(d) If $\chi = 1$ and A is an algebraically closed field then α_A^w is an isomorphism of algebraic varieties.

We note that $(\alpha_{A'}^w)_{A' \in \mathcal{C}}$ can be viewed as a polynomial family of injective maps $A'^n \rightarrow A'^n$ ($A' \in \mathcal{C}$, n as in 2.1). Hence the result follows from 1.2(ii).

4. APPLICATIONS

4.1. In this section we assume that A is an algebraically closed field. We write $G, U, {}^wU, U^w, T$ instead of $G_A, U_A, {}^wU_A, U_A^w, T_A$. By [L2, 4.11], G is naturally a connected reductive algebraic group over A with root datum \mathcal{R} and U is the unipotent radical of a Borel subgroup B^* of G with maximal torus T normalized by each \dot{s}_i . We assume that G is semisimple or equivalently that $\{i'; i \in I\}$ span a subgroup of finite index in X . Let δ be an automorphism of \mathcal{R} (necessarily of finite order, say c). The corresponding group automorphism $\delta : G \rightarrow G$ (see 2.10) preserves the algebraic group structure and has finite order c . Let \hat{G} be the semidirect product of G with the cyclic group of order c with generator d such that $dxd^{-1} = \delta(x)$ for all $x \in G$. Then \hat{G} is an algebraic group with identity component G . Let \mathcal{B} be the variety of Borel subgroups of G . For each $w \in W$ let \mathcal{O}_w be the set of all $(B, B') \in \mathcal{B} \times \mathcal{B}$ such that $B = xB^*x^{-1}$, $B' = x\dot{w}B^*\dot{w}^{-1}x^{-1}$ for some $x \in G$. We have $\mathcal{B} \times \mathcal{B} = \sqcup_{w \in W} \mathcal{O}_w$. As in [L5, 0.1, 0.2], for any $w \in W$ let

$$\mathfrak{B}_w = \{(g, B) \in Gd \times \mathcal{B}; (B, gBg^{-1}) \in \mathcal{O}_w\},$$

$$\tilde{\mathfrak{B}}_w = \{(g, g'^wU) \in Gd \times G/{}^wU; g'^{-1}gg' \in \dot{w}Ud\}.$$

Define $\pi_w : \tilde{\mathfrak{B}}_w \rightarrow \mathfrak{B}_w$ by $(g, g'^wU) \mapsto (g, g'B^*g'^{-1})$.

In the remainder of this section we assume that C is a δ -elliptic δ -conjugacy class in W and that $w \in C_{\min}$. Then π_w is a principal bundle with group $T_w = \{t_1 \in T; \dot{w}^{-1}t_1\dot{w} = dt_1d^{-1}\}$, a finite abelian group (see *loc.cit.*); the group T_w acts on $\tilde{\mathfrak{B}}_w$ by $t : (g, g'^wU) \mapsto (g, g't^{-1}wU)$. Now G acts on \mathfrak{B}_w by $x : (g, B) \mapsto (xgx^{-1}, xBx^{-1})$ and on $\tilde{\mathfrak{B}}_w$ by $x : (g, g'^wU) \mapsto (xgx^{-1}, xg'^wU)$. We show:

(a) Let \mathcal{O} be a G -orbit in $\tilde{\mathfrak{B}}_w$. There is a unique $v \in U^{\delta(w)}$ such that $(\dot{w}vd, {}^wU) \in \mathcal{O}$.

Clearly \mathcal{O} contains an element of the form $(\dot{w}ud, {}^wU)$ where $u \in U$. We first show the existence of v . It is enough to show that for some $z \in {}^wU, v \in U^{\delta(w)}$ we have $z\dot{w}udz^{-1} = \dot{w}vd$ that is $u = \dot{w}^{-1}z^{-1}\dot{w}v\delta(z)$. Setting $z' = \dot{w}^{-1}z^{-1}\dot{w}$, $w' = \delta(w)$ we see that it is enough to show that $u = z'v\dot{w}'\delta(z')^{-1}\dot{w}'^{-1}$ for some $z' \in \delta^{-1}(w')^{-1}U, v \in U^{w'}$. But this follows from 3.14(c) with $\chi = 1$ and w replaced by w' .

Now assume that $(\dot{w}vd, {}^wU) \in \mathcal{O}$, $(\dot{w}v'd, {}^wU) \in \mathcal{O}$ where $v, v' \in U^{\delta(w)}$. We have $\dot{w}v'd = u\dot{w}vdu^{-1}$ for some $u \in {}^wU$. Setting $u' = \dot{w}^{-1}u\dot{w} \in {}^{w^{-1}}U$ we have $v'd\dot{w} = u'v\dot{w}u'^{-1}$, that is $v' = u'v\delta(\dot{w})\delta(u'^{-1})\delta(\dot{w})^{-1}$. Using 3.14(b) (applied to $\delta(w)$ instead of w) we see that $u' = 1$ and $v = v'$. This completes the proof of (a).

We can reformulate (a) as follows.

(b) The closed subvariety $\{(\dot{w}vd, {}^wU); v \in U^{\delta(w)}\}$ of $\tilde{\mathfrak{B}}_w$ meets each G -orbit in

$\tilde{\mathfrak{B}}_w$ in exactly one point. Hence the space of G -orbits in $\tilde{\mathfrak{B}}_w$ can be identified with the affine space $U^{\delta(w)}$.

We show:

(c) *The closed subvariety $\{(\dot{w}vd, B^*); v \in U^{\delta(w)}\}$ of \mathfrak{B}_w is isomorphic to $U^{\delta(w)}$; its intersection with any G -orbit in \mathfrak{B}_w is a single T_w -orbit (for the restriction of the G -action), hence is a finite nonempty set.*

The first assertion of (c) is obvious. Now let $\bar{\mathcal{O}}$ be a G -orbit in \mathfrak{B}_w . Let

$$Z = \{(\dot{w}vd, B^*); v \in U^{\delta(w)}\} \cap \bar{\mathcal{O}}.$$

There exists a G -orbit \mathcal{O} in $\tilde{\mathfrak{B}}_w$ such that $\pi_w(\mathcal{O}) = \bar{\mathcal{O}}$. By (b) we can find $v \in U^{\delta(w)}$ such that $(\dot{w}vd, {}^wU) \in \mathcal{O}$. Then $(\dot{w}vd, B^*) = \pi_w(\dot{w}vd, {}^wU) \in \bar{\mathcal{O}}$ so that $Z \neq \emptyset$. If $(\dot{w}vd, B^*) \in Z$ and $t \in T_w$ then $(t\dot{w}vdt^{-1}, B^*) \in \bar{\mathcal{O}}$ and $(t\dot{w}vdt^{-1}, B^*) = (\dot{w}v''d, {}^wU)$ where

$$v'' = \dot{w}^{-1}t\dot{w}v'dt^{-1}d^{-1} = (dtd^{-1})v'(dt^{-1}d^{-1}) \in {}^wU.$$

Thus $(t\dot{w}vdt^{-1}, B^*) \in Z$ so that T_w acts on Z .

Assume now that $v, v' \in U^{\delta(w)}$ are such that $(\dot{w}vd, B^*) \in \bar{\mathcal{O}}$, $(\dot{w}v'd, B^*) \in \bar{\mathcal{O}}$. Then for some $x \in G$ we have

$$\pi_w(x\dot{w}vdx^{-1}, x{}^wU) = \pi_w(\dot{w}v'd, {}^wU).$$

Since π_w is a principal fibration with group T_w it follows that

$$(x\dot{w}vdx^{-1}, x{}^wU) = (\dot{w}v'd, t^{-1}{}^wU)$$

for some $t \in T_w$. Thus $(\dot{w}vd, {}^wU), (t\dot{w}v'dt^{-1}, {}^wU)$ are in the same G -orbit on $\tilde{\mathfrak{B}}_w$. Note that $(t\dot{w}v'dt^{-1}, {}^wU) = (\dot{w}v''d, {}^wU)$ where

$$v'' = \dot{w}^{-1}t\dot{w}v'dt^{-1}d^{-1} = (dtd^{-1})v'(dt^{-1}d^{-1}) \in {}^wU.$$

Using (b) we deduce that $v = v''$. Thus

$$(\dot{w}v'd, B^*) = (\dot{w}(\dot{w}^{-1}t^{-1}\dot{w})v(dtd^{-1})d, B^*) = (t^{-1}\dot{w}vdt, B^*) = t^{-1}(\dot{w}vd, B^*)$$

so that $(\dot{w}v'd, B^*), (\dot{w}vd, B^*)$ are in the same T_w -orbit. This completes the proof of (c).

We can reformulate (c) as follows.

(d) *The closed subvariety $\{(\dot{w}vd, B^*); v \in U^{\delta(w)}\}$ of \mathfrak{B}_w meets each G -orbit in \mathfrak{B}_w in exactly one T_w -orbit. Hence the space of G -orbits in \mathfrak{B}_w can be identified with the orbit space of the affine space $U^{\delta(w)}$ under an action of the finite group T_w .*

Statements like the last sentence in (b) and (d) were proved in [L5, 0.4(a)] assuming that G is almost simple of type A, B, C or D . The extension to exceptional types is new.

4.2. In this subsection we assume that $\delta = 1$ so that $d = 1$. Let γ be the unipotent class of G attached to C in [L3]. Recall from *loc.cit.* that γ has codimension $l(w)$ in G . The following result exhibits a closed subvariety of G isomorphic to the affine space $A^{l(w)}$ which intersects γ in a finite set.

(a) *The closed subvariety $\Sigma := \dot{w}U^w$ of G is isomorphic to U^w and $\Sigma \cap \gamma$ is a single T_w -orbit (for the conjugation action), hence is a finite nonempty set.*

According to [L4], the subset $\mathfrak{B}_w^\gamma = \{(g, B) \in \mathfrak{B}_w; g \in \gamma\}$ is a single G -orbit on \mathfrak{B}_w . Let $Z = \{(\dot{w}v, B^*); v \in U^w\} \cap \mathfrak{B}_w^\gamma$, $Z' = \{\dot{w}v; v \in U^w\} \cap \gamma$. The first projection defines a surjective map $Z \rightarrow Z'$. Since Z is a single T_w -orbit (see 3.1(c)), it follows that Z' is a single T_w -orbit. This proves (a).

4.3. In this subsection we assume that A, F_q, χ are as in 3.6(iv). Then $\pi = \delta\chi : G \rightarrow G$ is the Frobenius map for an F_q -rational structure on G . As in [DL], we set

$$\tilde{X}_w = \{g'^w U \in G/^w U; g'^{-1}\pi(g') \in \dot{w}U\}.$$

Now the finite group $G^\pi := \{g \in G; \pi(g) = g\}$ acts on \tilde{X}_w by $x : g'^w U \mapsto xg'^w U$. Let ${}^w U \backslash \backslash U$ be the set of orbits of the ${}^w U$ -action on U given by $u_1 : u \mapsto \dot{w}^{-1}u_1\dot{w}u\pi(u_1)^{-1}$. According to [DL, 1.12], we have a bijection $G^\pi \backslash \tilde{X}_w \xrightarrow{\sim} {}^w U \backslash \backslash U$, $g'^w U \mapsto \dot{w}^{-1}g'^{-1}\pi(g')$ with inverse induced by $u \mapsto g'^w U$ where $g' \in G$, $g'^{-1}\pi(g') = \dot{w}u$. Under the substitution $\dot{w}^{-1}u_1\dot{w} = u'$, the ${}^w U$ -action above on U becomes the ${}^{w^{-1}}U$ -action on U given by $u_2 : u \mapsto u'u\delta(\dot{w})p(u')^{-1}\delta(\dot{w})^{-1}$. Using 3.14(c) for $\delta(w)$ instead of w we see that the space of orbits of this action can be identified with $U^{\delta(w)}$. Thus we have the following result.

(a) *The space of orbits of G^π on \tilde{X}_w is quasi-isomorphic to the affine space $U^{\delta(w)}$.*

A statement like (a) was proved in [L5] assuming that G is almost simple of type A, B, C or D and $\delta = 1$. The extension to general G is new.

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